Sum of Powers of Roots

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Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation :

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0 \qquad , a_n \neq 0 \qquad \textbf{..... (1)}$$

We like to find the values of :

$$S_k = \alpha_1^k + \alpha_2^k + \ldots + \alpha_n^k \qquad , \qquad k = 1, 2, \ldots$$

and $$T_k = \alpha_1^{-k} + \alpha_2^{-k} + \ldots + \alpha_n^{-k} \qquad , \qquad k = 1, 2, \ldots$$

This problem can be solved by **Newton's formulas**, which can be found in most advanced level algebra books. However, Newton's formulas are difficult to derive and remember. This short article gives you an interesting alternative approach to get the solution.

Since $\alpha_1, \alpha_2, \ldots, \alpha_n$ are roots, therefore :

$$f(x) = a_n (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n) \qquad \textbf{..... (2)}$$

Taking natural logarithm of (2),

$$\ln|f(x)| = \ln|a_n| + \ln|x - \alpha_1| + \ln|x - \alpha_2| + \ldots + \ln|x - \alpha_n| \qquad \textbf{..... (3)}$$

Differentiate (3) with respect to x, we get:

$$\frac{f'(x)}{f(x)} = \frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \ldots + \frac{1}{x-\alpha_n} \qquad \textbf{..... (4)}$$

If the reader is not familiar with **logarithm differentiation**, simple product rule can be used to get (4).

Now apply the formula for **Infinite geometric series**: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ ($|x| < 1$) **..... (5)**

$$\frac{f'(x)}{f(x)} = \frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \ldots + \frac{1}{x-\alpha_n} = \frac{1}{x}\left\{\frac{1}{1-\frac{\alpha_1}{x}} + \frac{1}{1-\frac{\alpha_2}{x}} + \ldots + \frac{1}{1-\frac{\alpha_n}{x}}\right\}$$

$$= \frac{1}{x}\left\{\left[1 + \left(\frac{\alpha_1}{x}\right) + \left(\frac{\alpha_1}{x}\right)^2 + \left(\frac{\alpha_1}{x}\right)^3 + \ldots\right] + \left[1 + \left(\frac{\alpha_2}{x}\right) + \left(\frac{\alpha_2}{x}\right)^2 + \left(\frac{\alpha_2}{x}\right)^3 + \ldots\right]\right.$$

$$\left. + \ldots + \left[1 + \left(\frac{\alpha_n}{x}\right) + \left(\frac{\alpha_n}{x}\right)^2 + \left(\frac{\alpha_n}{x}\right)^3 + \ldots\right]\right\} \qquad \textbf{, by (5)}$$

$$= \frac{1}{x}\left\{n + \frac{1}{x}(\alpha_1 + \alpha_2 + \ldots + \alpha_n) + \frac{1}{x^2}\left(\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2\right) + \frac{1}{x^3}\left(\alpha_1^3 + \alpha_2^3 + \ldots + \alpha_n^3\right) + \ldots\right\}$$

$$= n\left(\frac{1}{x}\right) + S_1\left(\frac{1}{x}\right)^2 + S_2\left(\frac{1}{x}\right)^3 + S_3\left(\frac{1}{x}\right)^4 + \ldots$$

Result 1: S_k is the coefficient of $\left(\frac{1}{x}\right)^{k+1}$ in the expansion of $\frac{f'(x)}{f(x)}$, arranging in ascending power of $\frac{1}{x}$.

Similarly, $\dfrac{f'(x)}{f(x)} = \dfrac{1}{x-\alpha_1} + \dfrac{1}{x-\alpha_2} + \ldots + \dfrac{1}{x-\alpha_n}$

Therefore $-\dfrac{f'(x)}{f(x)} = \dfrac{1}{\alpha_1 - x} + \dfrac{1}{\alpha_2 - x} + \ldots + \dfrac{1}{\alpha_n - x} = \dfrac{1}{\alpha_1} \times \dfrac{1}{1-\dfrac{x}{\alpha_1}} + \dfrac{1}{\alpha_2} \times \dfrac{1}{1-\dfrac{x}{\alpha_2}} + \ldots + \dfrac{1}{\alpha_n} \times \dfrac{1}{1-\dfrac{x}{\alpha_n}}$

$$= \frac{1}{\alpha_1}\left\{1 + \left(\frac{x}{\alpha_1}\right) + \left(\frac{x}{\alpha_1}\right)^2 + \left(\frac{x}{\alpha_1}\right)^3 + \ldots\right\} + \frac{1}{\alpha_2}\left\{1 + \left(\frac{x}{\alpha_2}\right) + \left(\frac{x}{\alpha_2}\right)^2 + \left(\frac{x}{\alpha_2}\right)^3 + \ldots\right\}$$

$$+ \ldots + \frac{1}{\alpha_n}\left\{1 + \left(\frac{x}{\alpha_n}\right) + \left(\frac{x}{\alpha_n}\right)^2 + \left(\frac{x}{\alpha_n}\right)^3 + \ldots\right\} \qquad \textbf{, by (5)}$$

$$= \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \ldots + \frac{1}{\alpha_n}\right) + \left(\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \ldots + \frac{1}{\alpha_n^2}\right)x + \left(\frac{1}{\alpha_1^3} + \frac{1}{\alpha_2^3} + \ldots + \frac{1}{\alpha_n^3}\right)x^2 + \ldots$$

$$= T_1 + T_2\, x + T_3\, x^2 + \ldots$$

Result 2: T_k is the coefficient of x^{k-1} in the expansion of $-\dfrac{f'(x)}{f(x)}$, arranging in ascending powers of x.

Example 1

Let α, β be the roots of $x^2 + 2x + 4 = 0$, find

$S_1 = \alpha + \beta$, $S_2 = \alpha^2 + \beta^2$, $S_3 = \alpha^3 + \beta^3$, $S_4 = \alpha^4 + \beta^4$.

Solution

Let $f(x) = x^2 + 2x + 4$, $f'(x) = 2x + 2$.

$$\frac{f'(x)}{f(x)} = \frac{2x+2}{x^2+2x+4}$$

Using **Detached coefficient division**, with the first few steps shown:

(If you cannot see clearly the pattern of the variables, you may use long division.)

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              2 - 2 - 4 + 16 - 16
          ___________________________
1 + 2 + 4 ) 2 + 2
            2 + 4 + 8
            ---------
              - 2 - 8
              - 2 - 4 - 8
              -----------
                  - 4 + 8
                  - 4 - 8 - 16
                  ------------
                       16 + 16
                       16 + 32 + 64
                  -----------------
                           - 16 - 64
```

we get $\dfrac{f'(x)}{f(x)} = \dfrac{2x+2}{x^2+2x+4} = 2\left(\dfrac{1}{x}\right) - 2\left(\dfrac{1}{x}\right)^2 - 4\left(\dfrac{1}{x}\right)^3 + 16\left(\dfrac{1}{x}\right)^4 - 16\left(\dfrac{1}{x}\right)^5 + \ldots$

$\therefore$ By ***Result 1***, $S_1 = \alpha + \beta = -2$, $S_2 = \alpha^2 + \beta^2 = -4$, $S_3 = \alpha^3 + \beta^3 = 16$, $S_4 = \alpha^4 + \beta^4 = -16$

Exercise 1

Can you verify the above results using **Vieta's theorem**: $x^2 + 2x + 4 = 0$, $\alpha + \beta = -2$, $\alpha\beta = 4$?

Example 2

Let α, β, γ be the roots of the equation $x^3 - 2x^2 + x - 1 = 0$, find

$$T_1 = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \quad , \quad T_2 = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} \quad T_3 = \frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} \quad , \quad T_4 = \frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} .$$

Solution

Let $f(x) = x^3 - 2x^2 + x - 1$, $f'(x) = 3x^2 - 4x + 1$.

$$\frac{f'(x)}{f(x)} = \frac{3x^2 - 4x + 1}{x^3 - 2x^2 + x - 1} = -\frac{1 - 4x + 3x^2}{1 - x + 2x^2 - x^3} = -\frac{\left(\frac{1}{x}\right)^3 - 4\left(\frac{1}{x}\right)^2 + 3\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)^3 - \left(\frac{1}{x}\right)^2 + 2\left(\frac{1}{x}\right) - 1}$$

Using **Detached coefficient division**, with the first few steps shown:

(If you cannot see clearly the pattern of the variables, you may use long division.)

$$\begin{array}{r|l} & 1-3-2+5 \\ \hline 1-1+2-1 & 1-4+3 \\ & \underline{1-1+2-1} \\ & \quad -3+1+1 \\ & \quad \underline{-3+3-6+3} \\ & \qquad -2+7-3 \\ & \qquad \underline{-2+2-4+2} \\ & \qquad\quad 5+1-2 \end{array}$$

we get $\frac{f'(x)}{f(x)} = 1 - 3x - 2x^2 + 5x^3 + \ldots$

$\therefore$ By ***Result 2***, $T_1 = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1$, $T_2 = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = -3$

$T_3 = \frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} = -2$, $T_4 = \frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} = 5$

Exercise 2

Can you verify the above results using **Vieta's theorem** for the relation of roots and the polynomial equation:

$x^3 - 2x^2 + x - 1 = 0$, $\alpha + \beta + \gamma = 2$, $\alpha\beta + \beta\gamma + \gamma\alpha = 1$, $\alpha\beta\gamma = 1$?

(The verification may be long, but is a good algebra exercise.)